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Costruzioni in zona sismica

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SDOF systems: fundamentals

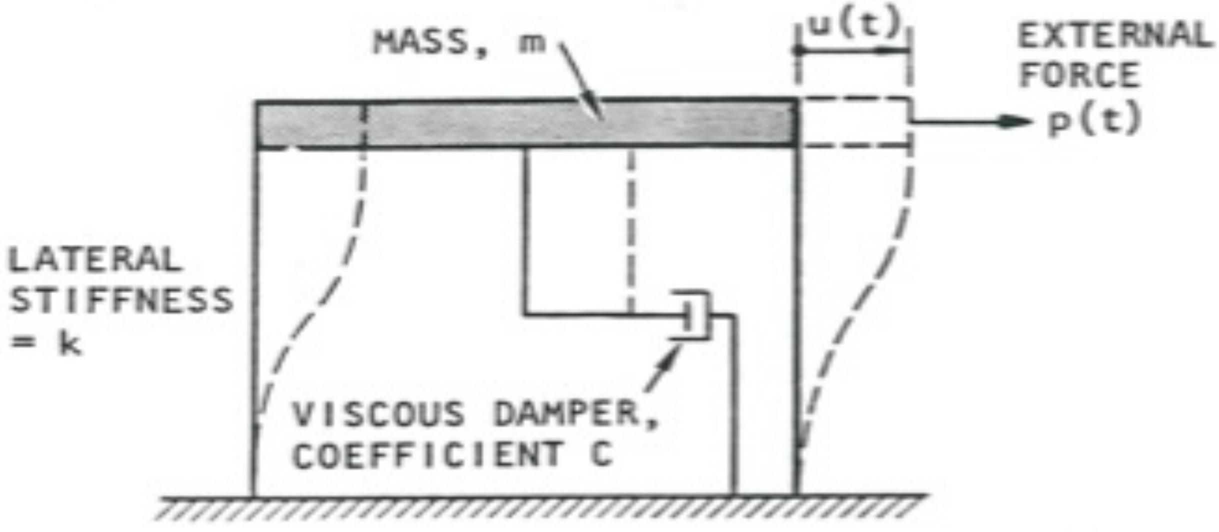
Problem statement

Given the mass m, stiffness k, damping c, and the excitation force p(t) or ground acceleration $\ddot{u}_g(t)$ a fundamental problem in structural dynamics is to determine the deformation response u(t) of the idealized one-story structure.

Other response quantities of interest such as base shear, can subsequently de determined from the deformation response

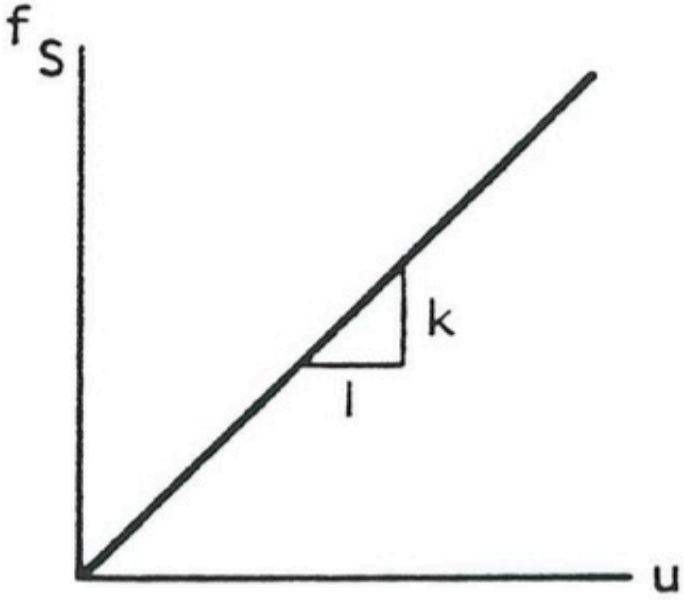
We will examine the response of the SDOF system in free vibration, to harmonic forces and to ground motion

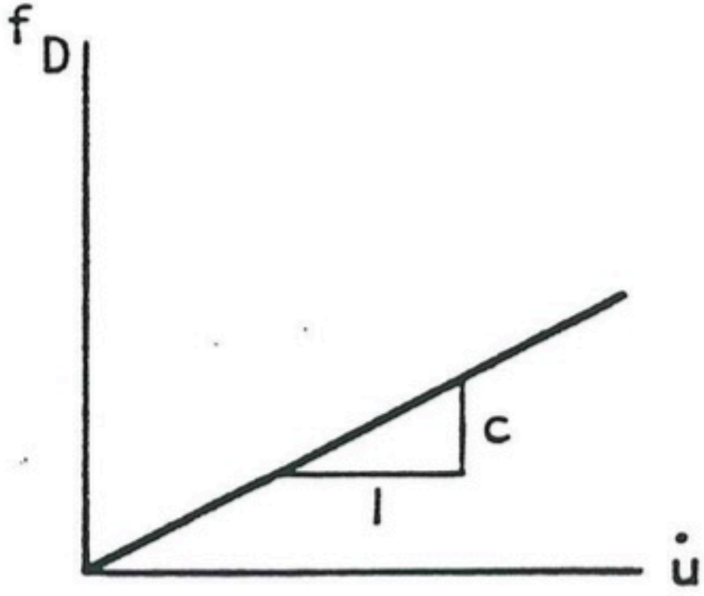
Dynamics of SDOF systems



(a) Idealized one-story structure

$$f_I + f_D + f_S = p(t)$$



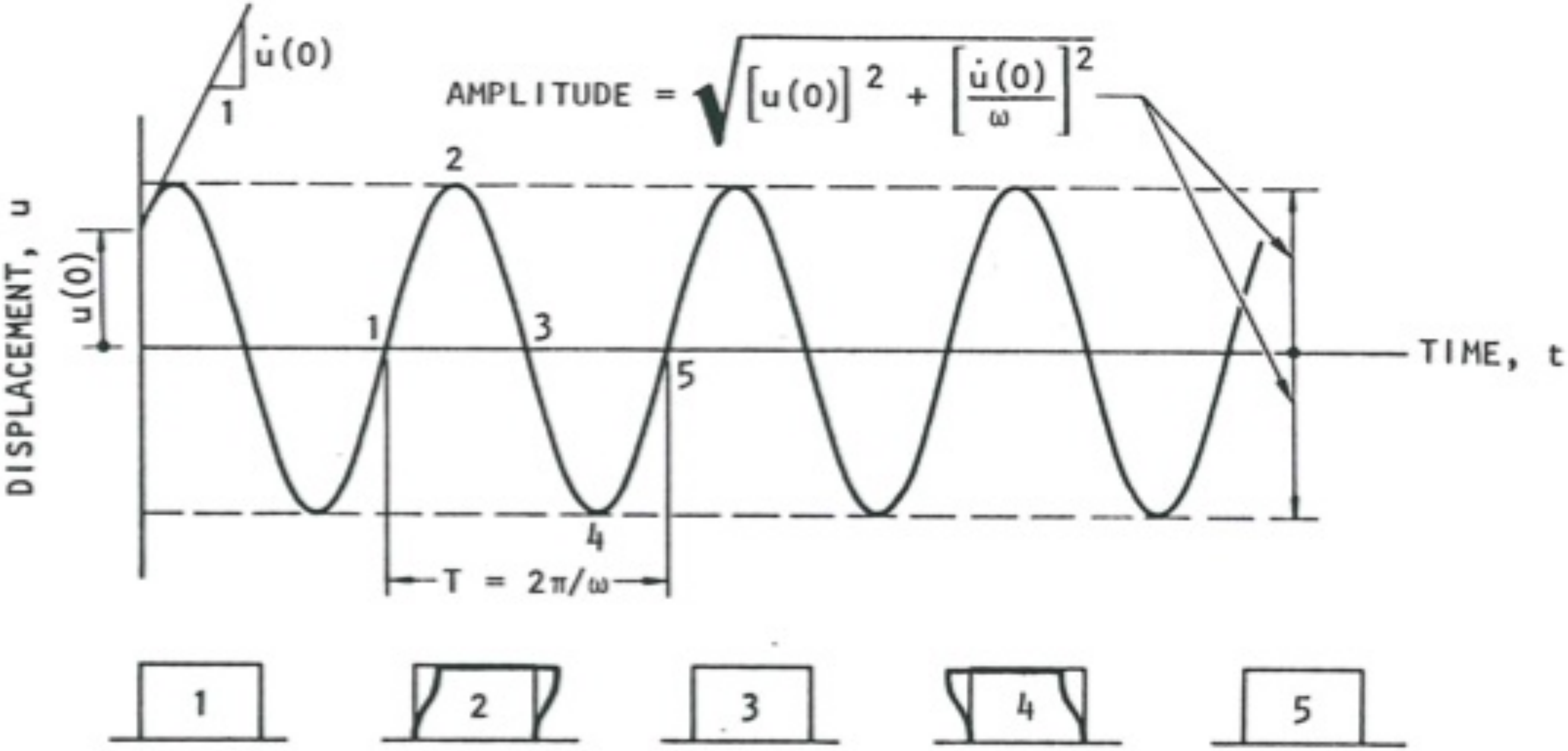


Elastic force-deformation relation	Inertia force	Damping force-velocity relation
$f_S = ku$	$f_I = m\ddot{u}$	$f_D = c\dot{u}$

Equation of motion

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

Free vibration response: undamped structures



Equation of motion in canonic form

$$\ddot{u}(t) + \omega^2 u(t) = 0, \qquad \omega = \sqrt{\frac{k}{m}}$$

Explicit solution

$$u(t) = \frac{\dot{u}_0}{\omega}\sin \omega t + u_0 \cos \omega t$$

Free vibration response: natural period

Natural circular frequency of vibration $\omega \ (rad/\text{sec})$

Natural period of vibration T (sec) $\text{T} = \dfrac{2\pi}{\omega}$

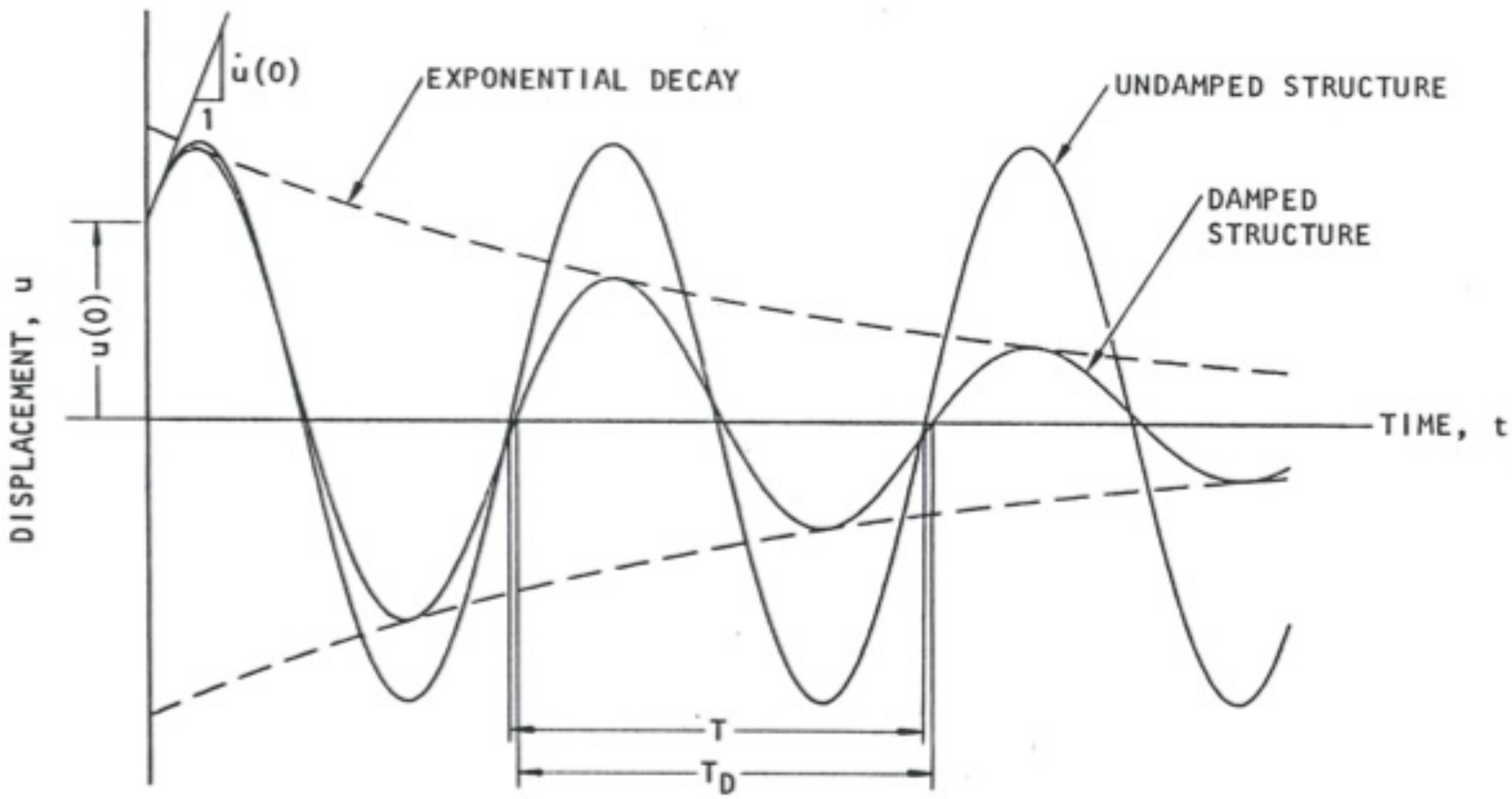
Natural frequency of vibration $f \ (Hz)$ $f = \dfrac{1}{\text{T}} = \dfrac{\omega}{2\pi}$

The term natural is used to emphasize the fact that these are natural properties of the structure

$$\omega = \sqrt{\frac{k}{m}}$$

The free vibration properties only depend on the mass and the stiffness of the structure

Free vibration response: damped structures



$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = 0$$

$$\omega_D = \omega\sqrt{1-\xi^2}$$

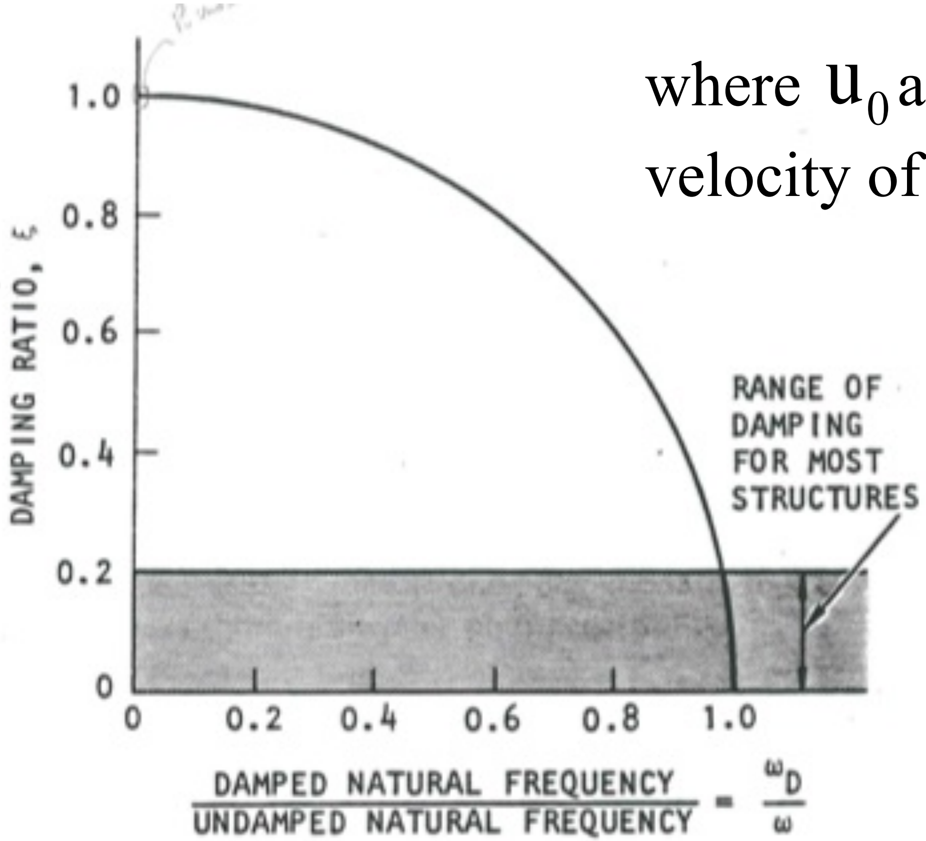
Damping ratio: $\xi = \dfrac{c}{2m\omega} = \dfrac{c}{2\sqrt{km}}$

$$T_D = \frac{T}{\sqrt{1-\xi^2}}$$

Free vibration response: damped structures

$$u(t) = e^{-\xi\,\omega\,t}\left[u_0\left(\cos\omega_D t + \frac{\xi\omega}{\omega_D}\sin\omega_D t\right) + \frac{\dot{u}_0}{\omega_D}\sin\omega_D t\right]$$

$$= Ae^{-\xi\,\omega\,t}\cos(\omega_D t - \theta)$$



where u_0 and $\dot{u}_0$ are initial displacement and velocity of the SDOF system

θ is the angular phase shift

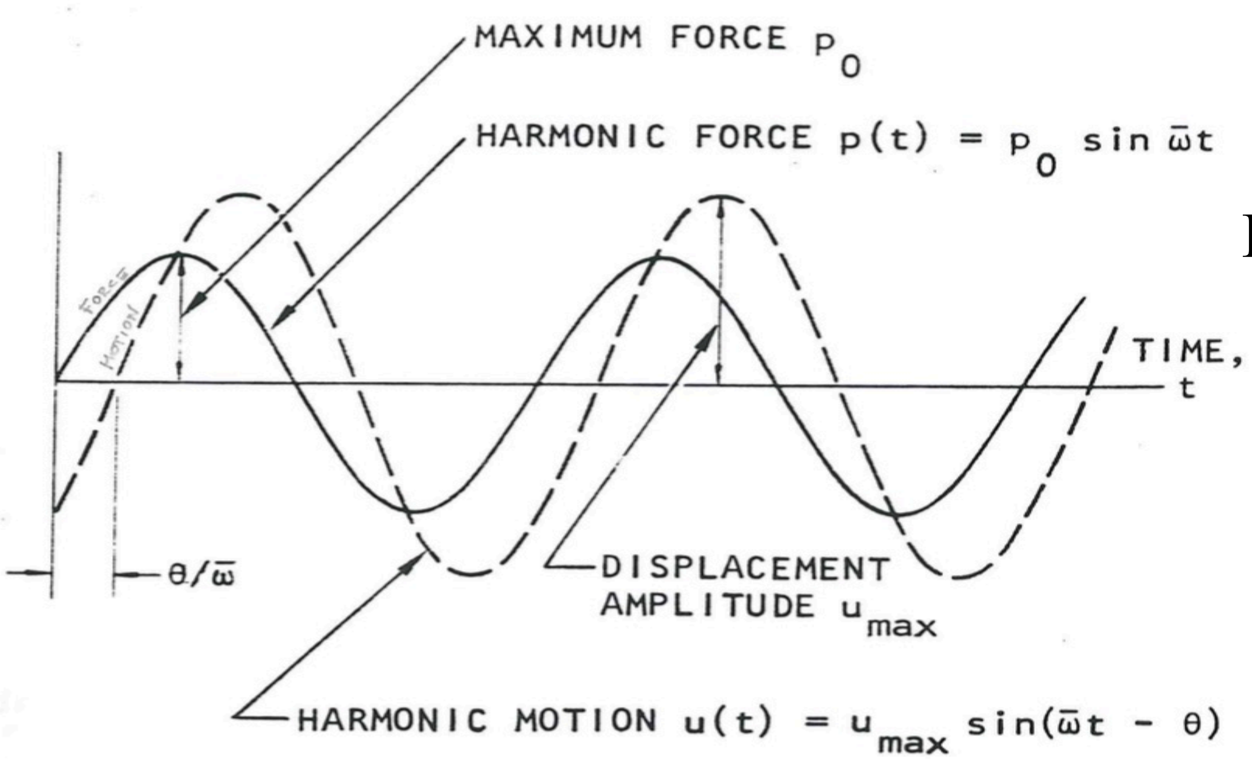
$$\omega_D = \omega\sqrt{1-\xi^2}$$

Response to harmonic exitation

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = \frac{p_0}{m}\sin\omega_f t$$

$$u(t) = \boxed{Ae^{-\xi\omega t}\cos(\omega_D t - \theta)} + \boxed{Ru_{st}\sin(\omega_f t - \theta_f)}$$

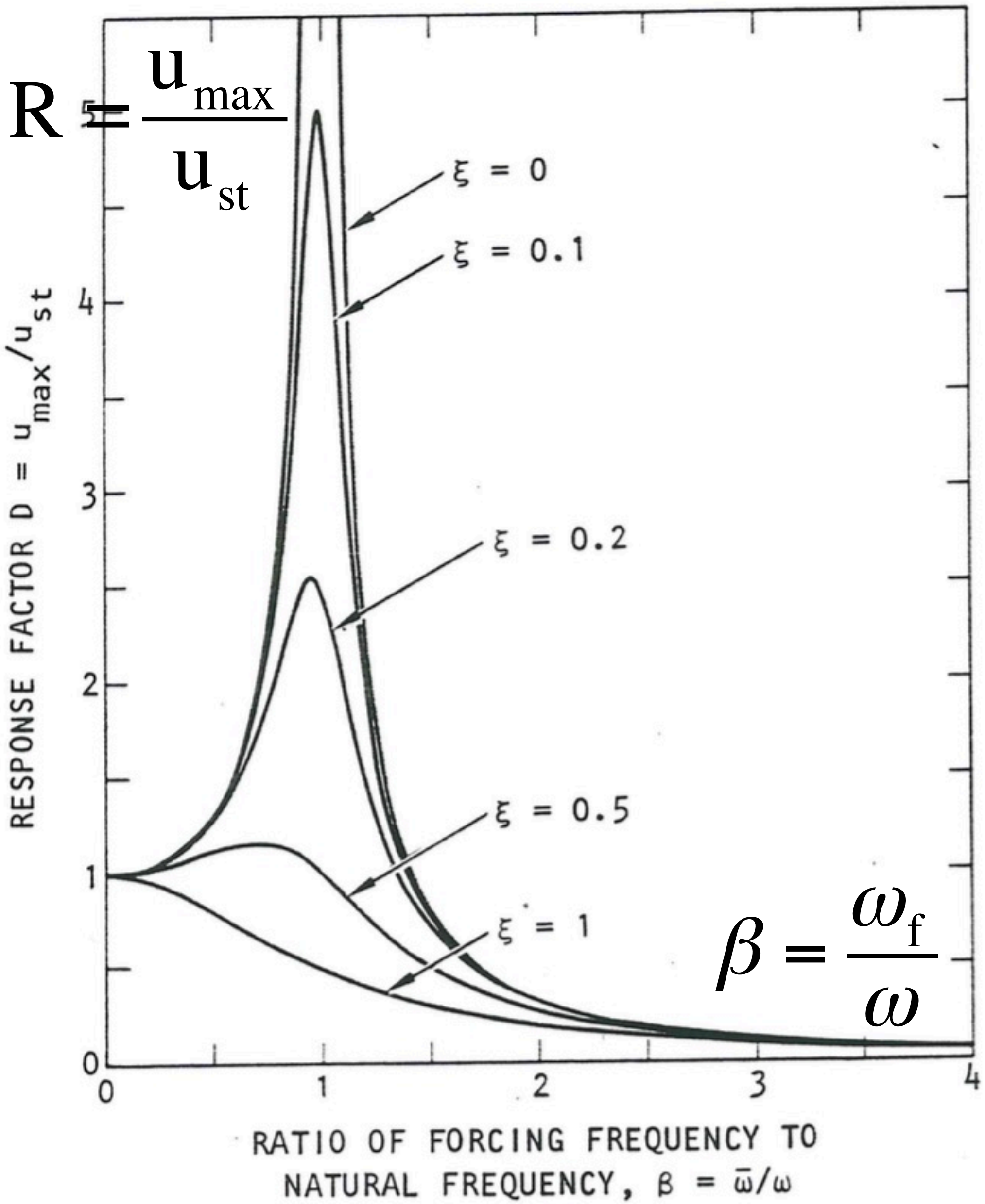
Free vibration transient motion — Steady state motion



Response factor

$$R = \frac{u_{max}}{u_{st}}$$

Response to harmonic exitation



$$\beta = \frac{\omega_f}{\omega} \qquad u_{st} = \frac{p_0}{k}$$

Response factor $\quad R = \dfrac{u_{max}}{u_{st}}$

(dynamic to static response amplitude)

$$R = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\xi\beta)^2}}$$

Angular phase shift

$$\theta_f = \tan^{-1}\left(\frac{2\xi\beta}{1-\beta^2}\right)$$

Response to harmonic exitation

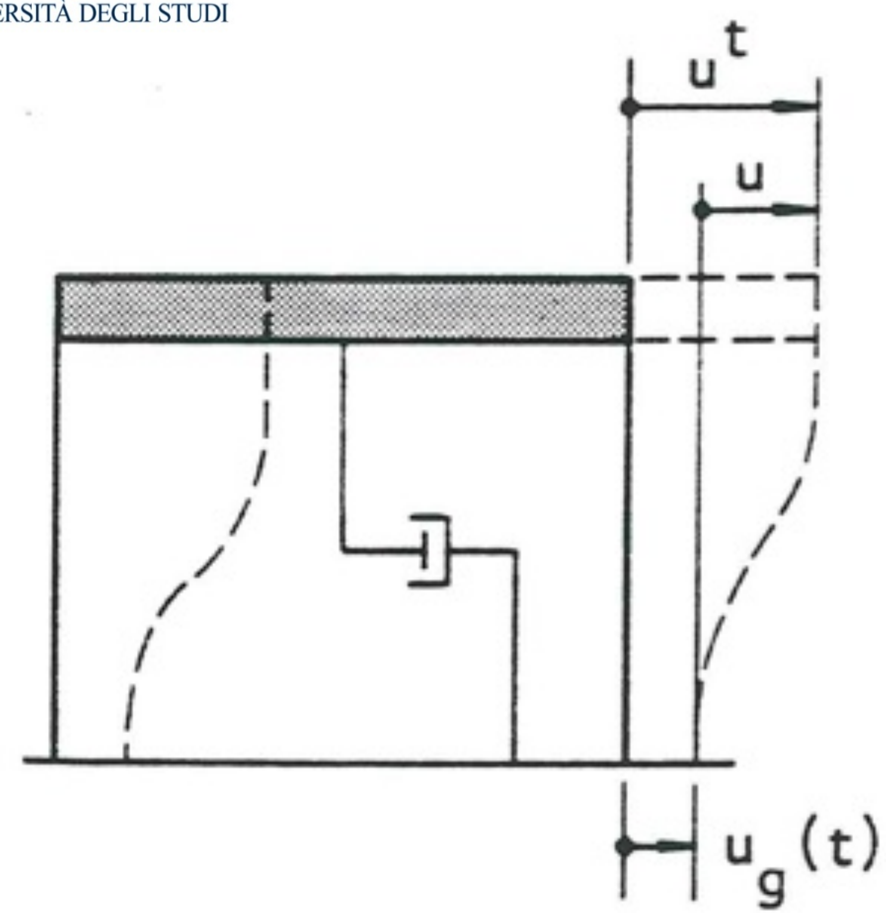
For small values of β the maximum displacement is controlled by the stiffness of the system with little effect of mass or damping.

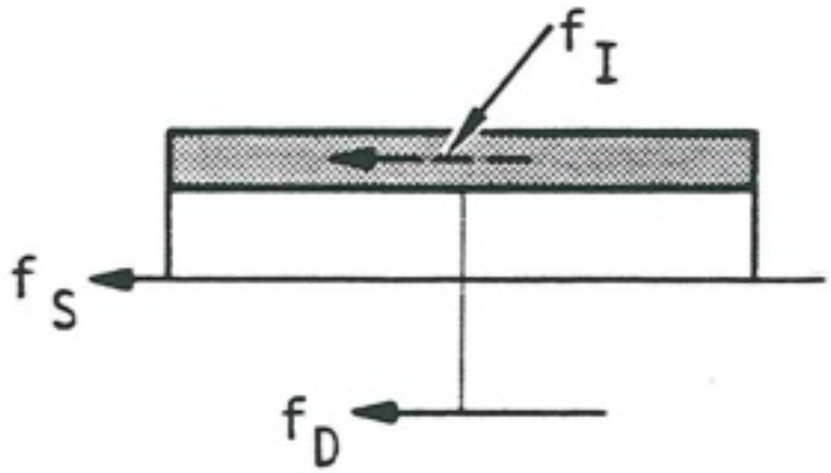
For β clos to 1, the response factor is about $R=1/2\xi$, that is the response factor is inversely proportional to the damping ratio, with negligible influence of mass or stiffness.

The response factor is essentially independent of damping and approaches to zero as the forcing frequency ω_f becomes much higher than the natural frequency ω of the structure.

It can be shown that at high forcing frequencies the maximum displacement depends primarily on the mass.

Earthquake ground motion





Total displacement

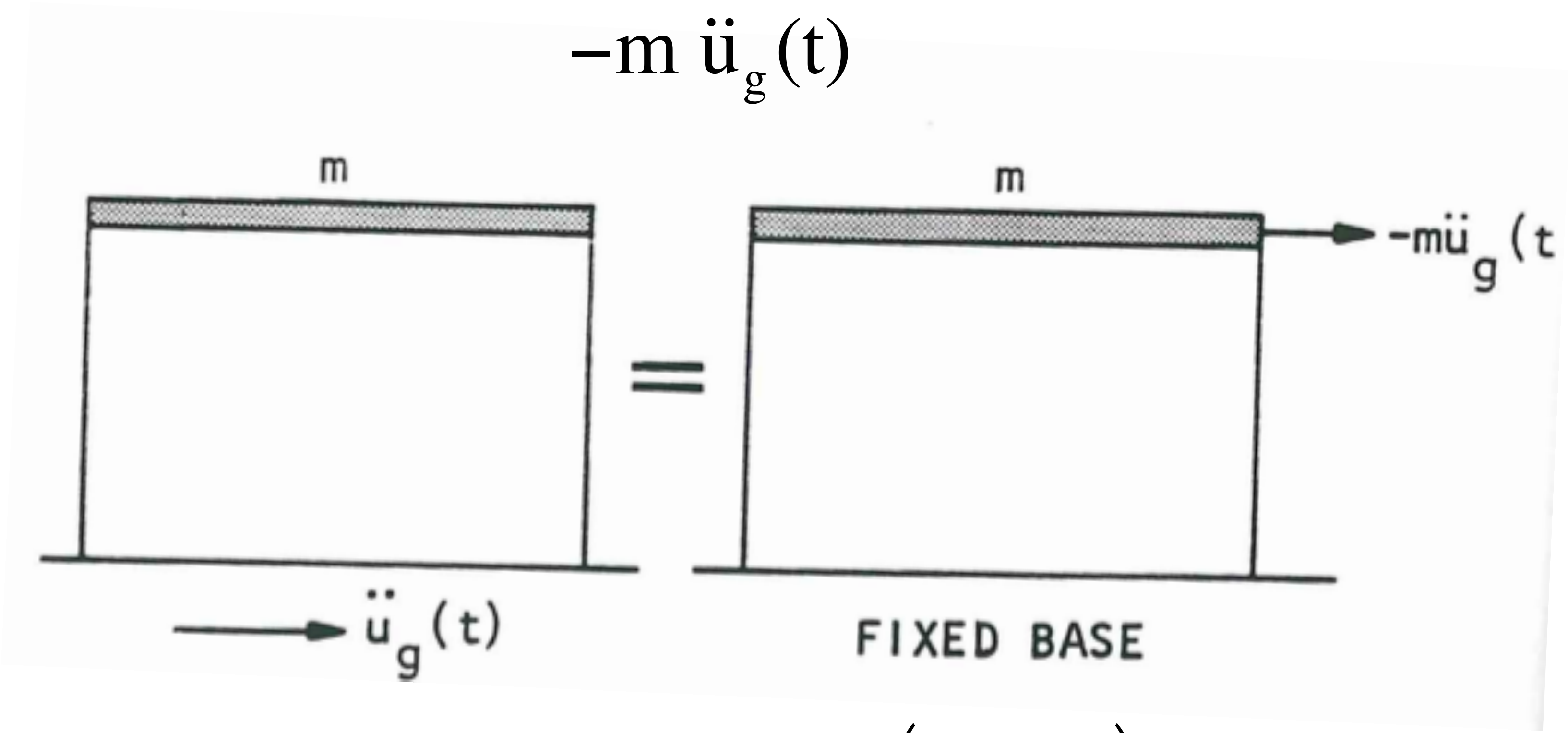
$$u^t = u_g + u$$

Equation of dynamic equilibrium

$$f_I + f_D + f_S = 0$$

Equivalence between ground motion and effective force

$$-m\,\ddot{u}_g(t)$$



$$f_I = m\,\ddot{u}^t \qquad f_I = m\left(\ddot{u}_g + \ddot{u}\right)$$

Equation of motion

$$m\ddot{u} + c\dot{u} + ku = -\,m\,\ddot{u}_g(t)$$

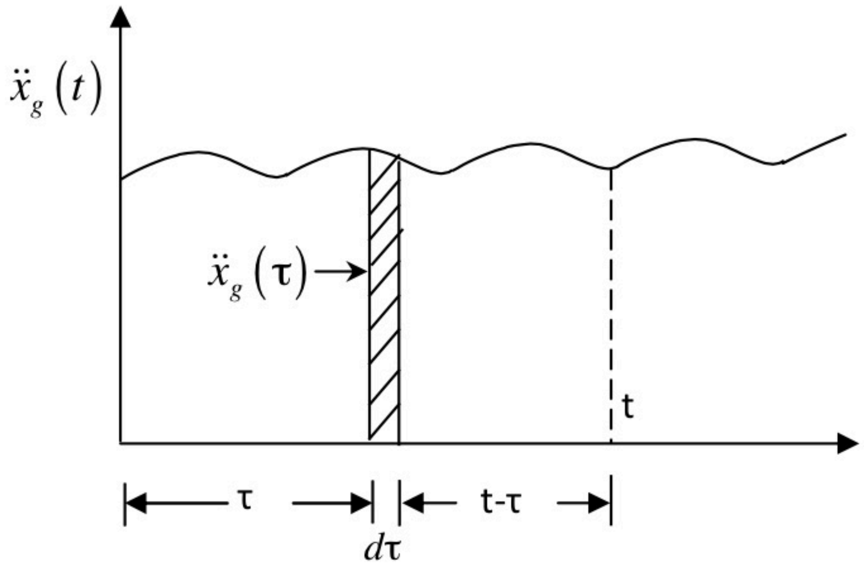
Response to earthquake ground motion

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = -\ddot{u}_g(t)$$

The solution leads to the deformation response u(t) which depends on the characteristics of ground acceleration $\ddot{u}_g(t)$, the natural circular frequency of vibration $\omega=(k/m)^{1/2}$ (or equivalently the natural period of vibration T) of the structure and the damping ratio ξ of the structure.

Earthquake ground accelerations vary irregularly to such an extent that analytical evaluation of this integral must be ruled out.

Let us assume that the irregular ground acceleration is made up of very brief impulses. The vibration caused by



$$u(t) = \dot{u}_0\, h(t)$$

$$h(t) = \frac{e^{-\xi\,\omega\, t}}{\omega_D} \sin \omega_D t$$

Where $\dot{u}_0$ is the initial velocity.

The earthquake accelerogram is digitized and appropriately filtered to control accelerogram errors and baseline distortion.
For instance, the accelerogram could be defined at 0.02 second time intervals. With the ground acceleration defined in this manner the response history could be determined by numerical evaluation of the Duhamel integral.

It should be noted that the small changes in velocity and displacement occurring during the time interval $d\tau$ will make a negligible contribution to the change in momentum. The change in velocity during the interval is

$$d\dot{u}(t) = -\ddot{u}_g(\tau)d\tau \tag{3.13}$$

Thus, the change in displacement at time, t caused by the impulse at τ is given by

$$du(t) = -\ddot{u}_g(\tau)d\tau\ h(t - \tau) \tag{3.14}$$

Each impulse in Figure (3.3) will produce a vibration of this form. Because the system is linear, the effect of each impulse is independent of every other impulse and the total resulting motion can be obtained by the principle of super position.

$$u(t) = -\int_0^t \ddot{u}_g(\tau)\ h(t - \tau)d\tau \tag{3.15}$$

This integral is known as **convolution or Duhamel integral**. Explicit solution may be obtained for simple forms of forcing function such as rectangular and triangular.